

FINITE SIMPLE GROUPS AND LOCALIZATION

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ABSTRACT

The purpose of this paper is to explore the concept of localization, which comes from homotopy theory, in the context of finite simple groups. We give an easy criterion for a finite simple group to be a localization of some simple subgroup and we apply it in various cases. Iterating this process allows us to connect many simple groups by a sequence of localizations. We prove that all sporadic simple groups (except possibly the Monster) and several groups of Lie type are connected to alternating groups. The question remains open whether or not there are several connected components within the family of finite simple groups.

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Introduction

The concept of localization plays an important role in homotopy theory. The introduction by Bousfield of homotopical localization functors in [2] and more recently its popularization by Farjoun in [8] has led to the study of localization functors in other categories. Special attention has been set on the category of groups Gr , as the effect of a homotopical localization on the fundamental group is often best described by a localization functor $L: Gr \rightarrow Gr$.

A **localization functor** is a pair (L, η) consisting of a functor $L: Gr \rightarrow Gr$ together with a natural transformation $\eta: Id \rightarrow L$ from the identity functor, such that L is idempotent, meaning that the two morphisms $\eta_{LG}, L(\eta_G): LG \rightarrow LLG$ coincide and are isomorphisms. A group homomorphism $\varphi: H \rightarrow G$ is called in turn a **localization** if there exists a localization functor (L, η) such that $G = LH$ and $\varphi = \eta_H: H \rightarrow LH$ (but we note that the functor L is not uniquely determined by φ). In this situation, we often say that G is a localization of H . A very simple characterization of localizations can be given without mentioning localization functors: A group homomorphism $\varphi: H \rightarrow G$ is a localization if and only if φ induces a bijection

$$(0.1) \quad \varphi^*: \text{Hom}(G, G) \cong \text{Hom}(H, G)$$

as mentioned in [4, Lemma 2.1]. In the last decade several authors (Casacuberta, Farjoun, Libman, Rodríguez) have directed their efforts towards deciding which algebraic properties are preserved under localization. An exhaustive survey about this problem is nicely exposed in [4] by Casacuberta. For example, any localization of an abelian group is again abelian. Similarly, nilpotent groups of class at most 2 are preserved, but the question remains open for arbitrary nilpotent groups. Finiteness is not preserved, as shown by the example $A_n \rightarrow SO(n-1)$ for $n \geq 10$ (this is the main result in [18]). In fact, it has been shown by Göbel and Shelah in [10] that any non-abelian finite simple group has arbitrarily large localizations (a previous version of this result, assuming the generalized continuum hypothesis, was obtained in [9]). In particular it is not easy to determine all possible localizations of a given object. Thus we restrict ourselves to the study of finite groups and wonder if it would be possible to understand the finite localizations of a given finite simple group. This paper is a first step in this direction.

Libman [19] observed recently that the inclusion $A_n \hookrightarrow A_{n+1}$ of alternating groups is a localization if $n \geq 7$. His motivation was to find a localization where new torsion elements appear (e.g., $A_{10} \hookrightarrow A_{11}$ is such a localization since A_{11} contains elements of order 11). In these examples, the groups are simple, which

simplifies considerably the verification of formula (0.1). It suffices to check if $\text{Aut}(G) \cong \text{Hom}(H, G) - \{0\}$.

This paper is devoted to the study of the behaviour of injective localizations with respect to simplicity. We first give a criterion for an inclusion of a simple group in a finite simple group to be a localization. We then find several infinite families of such localizations, for example $L_2(p) \hookrightarrow A_{p+1}$ for any prime $p \geq 13$ (cf. Proposition 2.3). Here $L_2(p) = PSL_2(p)$ is the projective special linear group. It is striking to notice that the three conditions that appear in our criterion for an inclusion of simple groups $H \hookrightarrow G$ to be a localization already appeared in the literature. For example, the main theorem of [17] states exactly that $J_3 \hookrightarrow E_6(4)$ is a localization (see Section 3). Similarly the main theorem in [24] states that $Sz(32) \hookrightarrow E_8(5)$ is a localization. Hence the language of localization theory can be useful to shortly reformulate some rather technical properties.

By Libman’s result, the alternating groups A_n , for $n \geq 7$, are all connected by a sequence of localizations. We show that $A_5 \hookrightarrow A_6$ is also a localization. A more curious way allows us to connect A_6 to A_7 by a zigzag of localizations:

$$A_6 \hookrightarrow T \hookrightarrow Ru \leftrightarrow L_2(13) \hookrightarrow A_{14} \leftrightarrow \dots \leftrightarrow A_7$$

where T is the Tits group, and Ru the Rudvalis group. This yields the concept of rigid component of a simple group. The idea is that among all inclusions $H \hookrightarrow G$, those that are localizations deserve our attention because of the “rigidity condition” imposed by (0.1): Any automorphism of G is completely determined by its restriction to H . So, we say that two groups H and G lie in the same **rigid component** if H and G can be connected by a zigzag of inclusions which are all localizations.

Many finite simple groups can be connected to the alternating groups. Here is our main result:

THEOREM: *The following finite simple groups all lie in the same rigid component:*

- (i) All alternating groups A_n ($n \geq 5$).
- (ii) The Chevalley groups $L_2(q)$ where q is a prime power ≥ 5 .
- (iii) The Chevalley groups $U_3(q)$ where q is any prime power.
- (iv) The Chevalley groups $G_2(p)$ where p is an odd prime such that $(p+1, 3) = 1$.
- (v) All sporadic simple groups, except possibly the Monster.
- (vi) The Chevalley groups $L_3(3), L_3(5), L_3(11), L_4(3), U_4(2), U_4(3), U_5(2), U_6(2), S_4(4), S_6(2), S_8(2), D_4(2), {}^2D_4(2), {}^2D_5(2), {}^3D_4(2), D_4(3), G_2(2)', G_2(4), G_2(5), G_2(11), E_6(4), F_4(2)$, and $T = {}^2F_4(2)'$.

The proof is an application of the localization criteria which are given in Sections 1 and 2, but requires a careful checking in the ATLAS [5], or in the more complete papers about maximal subgroups of finite simple groups (e.g. [14], [21], [27]). We do not know if the Monster can be connected to the alternating groups.

It is still an open problem to know how many rigid components of finite simple groups there are, even though our main theorem seems to suggest that there is only one. We note that the similar question for non-injective localizations has a trivial answer (see Section 1).

Let us finally mention that simplicity is not necessarily preserved by localization. This is the subject of the separate paper [22], where we exhibit, for example, a localization map from the Mathieu group M_{11} to the double cover of the Mathieu group M_{12} . In our context this implies that the rigid component of a simple group may contain a non-simple group. This answers negatively a question posed both by Libman in [19] and Casacuberta in [4] about the preservation of simplicity. In these papers it was also asked whether perfectness is preserved, but we leave this question unsolved.

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1. A localization criterion

Let us fix from now on a finite simple group G . In Theorem 1.4 below we list necessary and sufficient conditions for an inclusion $H \hookrightarrow G$ between two non-abelian finite simple groups to be a localization. These conditions are easier to deal with if the groups H and G satisfy some extra assumptions, as we show in the corollaries after the theorem. The proof is a variation of that of Corollary 4 in [9].

We note here that we only deal with injective group homomorphisms because non-injective localizations abound. For example, for any two finite groups G_1 and G_2 of coprime orders, $G_1 \times G_2 \rightarrow G_1$ and $G_1 \times G_2 \rightarrow G_2$ are localizations. So the analogous concept of rigid component defined using non-injective localizations has no interest, since obviously any two finite groups are in the same component.

If the inclusion $i: H \hookrightarrow G$ is a localization, then so is the inclusion $H' \hookrightarrow G$ for any subgroup H' of G which is isomorphic to H . This shows that the choice of the subgroup H among isomorphic subgroups does not matter.

Let $c: G \rightarrow \text{Aut}(G)$ be the natural injection of G defined as $c(g) = c_g: G \rightarrow G$, where c_g is the inner automorphism given by $x \mapsto gxg^{-1}$. We shall always identify in this way a simple group G with a subgroup of $\text{Aut}(G)$ and the quotient $\text{Out}(G) = \text{Aut}(G)/G$ is then called the **group of outer automorphisms** of G .

LEMMA 1.1: *Let G be a non-abelian simple group. Then the following diagram commutes*

$$\begin{array}{ccc}
 G & \xrightarrow{\alpha} & G \\
 c \downarrow & & \downarrow c \\
 \text{Aut}(G) & \xrightarrow{c_\alpha} & \text{Aut}(G)
 \end{array}$$

for any automorphism $\alpha \in \text{Aut}(G)$.

Proof: This is a trivial check. ■

LEMMA 1.2: *Let H be a non-abelian simple subgroup of a finite simple group G . Suppose that the inclusion $i: H \hookrightarrow G$ extends to an inclusion of their automorphism groups $i: \text{Aut}(H) \hookrightarrow \text{Aut}(G)$, i.e., the following diagram commutes*

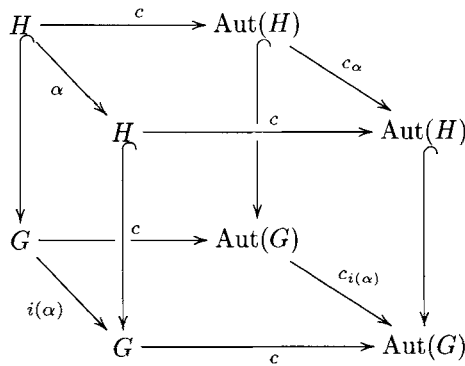
$$\begin{array}{ccc}
 H & \xrightarrow{i} & G \\
 c \downarrow & & \downarrow c \\
 \text{Aut}(H) & \xrightarrow{i} & \text{Aut}(G)
 \end{array}$$

Then every automorphism $\alpha: H \rightarrow H$ extends to an automorphism $i(\alpha): G \rightarrow G$.

Proof: We have to show that the following square commutes:

$$\begin{array}{ccc}
 H & \xrightarrow{\alpha} & H \\
 i \downarrow & & \downarrow i \\
 G & \xrightarrow{i(\alpha)} & G
 \end{array}$$

To do so we consider this square as the left-hand face of the cubical diagram



The top and bottom squares commute by Lemma 1.1. The front and back squares are the same and commute by assumption. The right-hand square commutes as well because i is a homomorphism. This forces the left-hand square to commute and we are done. ■

Remark 1.3: As shown by the preceding lemma, it is stronger to require that the inclusion $i: H \hookrightarrow G$ extends to an inclusion $i: \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ than to require that every automorphism of H extends to an automorphism of G . In general we have an exact sequence

$$1 \rightarrow C_{\text{Aut}(G)}(H) \rightarrow N_{\text{Aut}(G)}(H) \rightarrow \text{Aut}(H)$$

so the second condition is equivalent to the fact that this is a short exact sequence. However, in the presence of the condition $C_{\text{Aut}(G)}(H) = 1$, which plays a central role in this paper, we find that $N_{\text{Aut}(G)}(H) \cong \text{Aut}(H)$. Thus any automorphism of H extends to a unique automorphism of G , and this defines a homomorphism $i: \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ extending the inclusion $H \hookrightarrow G$. Therefore, if the condition $C_{\text{Aut}(G)}(H) = 1$ holds, we have a converse of the above lemma and both conditions are equivalent. We will use the first in the statements of the following results, even though it is the stronger one. It is indeed easier to check in the applications.

THEOREM 1.4: *Let H be a non-abelian simple subgroup of a finite simple group G and let $i: H \hookrightarrow G$ be the inclusion. Then i is a localization if and only if the following three conditions are satisfied:*

1. *The inclusion $i: H \hookrightarrow G$ extends to an inclusion $i: \text{Aut}(H) \hookrightarrow \text{Aut}(G)$.*
2. *Any subgroup of G which is isomorphic to H is conjugate to H in $\text{Aut}(G)$.*

3. The centralizer $C_{\text{Aut}(G)}(H) = 1$.

Proof: If i is a localization, all three conditions have to be satisfied. Indeed given an automorphism $\alpha \in \text{Aut}(H)$, formula (0.1) tells us that there exists a unique group homomorphism $\beta: G \rightarrow G$ such that $\beta \circ i = i \circ \alpha$. Since G is finite and simple, β is an automorphism. We set then $i(\alpha) = \beta$ and condition (1) follows. Given a subgroup $j: H' \hookrightarrow G$ and an isomorphism $\phi: H \rightarrow H'$, a similar argument with formula (0.1) ensures the existence of an automorphism $\beta \in \text{Aut}(G)$ such that $\beta \circ i = j \circ \phi$. Thus condition (2) holds. Finally, condition (3) is also valid since the unique extension of 1_H to G is the identity.

Assume now that all three conditions hold. For any given homomorphism $\varphi: H \rightarrow G$, we need a unique homomorphism $\Phi: G \rightarrow G$ such that $\Phi \circ i = \varphi$. If φ is trivial, we choose of course the trivial homomorphism $\Phi: G \rightarrow G$. It is unique since H is in the kernel of Φ , which must be equal to G by simplicity. Hence, we can suppose that φ is not trivial. Since H is simple we have that $\varphi(H) \leq G$ and $H \cong \varphi(H)$.

By (2) there is an automorphism $\alpha \in \text{Aut}(G)$ such that $c_\alpha(\varphi(H)) = H$, or equivalently by Lemma 1.1, $\alpha(\varphi(H)) = H$. Therefore the composite map

$$H \xrightarrow{\varphi} \varphi(H) \xrightarrow{\alpha|_{\varphi(H)}} H$$

is some automorphism β of H . By condition (1) this automorphism of H extends to an automorphism $i(\beta): G \rightarrow G$. That is, the following square commutes:

$$\begin{array}{ccc} H & \xrightarrow{\beta} & H \\ \downarrow i & & \downarrow i \\ G & \xrightarrow{i(\beta)} & G \end{array}$$

The homomorphism $\Phi = \alpha^{-1}i(\beta)$ extends φ as desired. We prove now it is unique. Suppose that $\Phi': G \rightarrow G$ is a homomorphism such that $\Phi' \circ i = \varphi$. Then, since G is simple, $\Phi' \in \text{Aut}(G)$. The composite $\Phi^{-1}\Phi'$ is an element in the centralizer $C_{\text{Aut}(G)}(H)$, which is trivial by (3). This finishes the proof of the theorem. ■

Remark 1.5: The terminology used for condition (2) is that two subgroups H and H' in G **fuse** in $\text{Aut}(G)$ if there is an automorphism $\alpha \in \text{Aut}(G)$ such that $\alpha(H) = H'$. Assuming condition (1) in Theorem 1.4 we have a short exact sequence

$$1 \rightarrow C_{\text{Aut}(G)}(H) \rightarrow N_{\text{Aut}(G)}(H) \rightarrow \text{Aut}(H) \rightarrow 1$$

(compare with Remark 1.3). Moreover there are $|\text{Aut}(G)| / |N_{\text{Aut}(G)}(H)|$ subgroups in the conjugacy class of H in $\text{Aut}(G)$ and $|G| / |N_G(H)|$ subgroups in the conjugacy class of H in G . If condition (2) also holds, this implies that the total number of conjugacy classes of subgroups isomorphic to H in G is equal to

$$\frac{|\text{Aut}(G)|}{|N_{\text{Aut}(G)}(H)|} \cdot \frac{|N_G(H)|}{|G|} = \frac{|\text{Out}(G)| \cdot |G|}{|\text{Aut}(H)| \cdot |C_{\text{Aut}(G)}(H)|} \cdot \frac{|N_G(H)|}{|G|}.$$

Condition (3) is thus equivalent to the following one, which is sometimes easier to verify:

3'. *The number of conjugacy classes of subgroups of G isomorphic to H is equal to*

$$\frac{|\text{Out}(G)|}{|\text{Out}(H)|} \cdot \frac{|N_G(H)|}{|H|}.$$

We obtain immediately the following corollaries. Using the terminology in [23, p. 158], recall that a group is **complete** if it has trivial centre and every automorphism is inner.

COROLLARY 1.6: *Let H be a non-abelian simple subgroup of a finite simple group G and let $i: H \hookrightarrow G$ be the inclusion. Assume that H and G are complete groups. Then i is a localization if and only if the following two conditions are satisfied:*

1. Any subgroup of G which is isomorphic to H is conjugate to H .
2. $C_G(H) = 1$.

The condition $C_G(H) = 1$ is here equivalent to $N_G(H) = H$. This is often easier to check. It is in particular always the case when H is a maximal subgroup of G . This leads us to the next corollary.

COROLLARY 1.7: *Let H be a non-abelian simple subgroup of a finite simple group G and let $i: H \hookrightarrow G$ be the inclusion. Assume that H is a maximal subgroup of G . Then i is a localization if and only if the following three conditions are satisfied:*

1. The inclusion $i: H \hookrightarrow G$ extends to an inclusion $i: \text{Aut}(H) \hookrightarrow \text{Aut}(G)$.
2. Any subgroup of G which is isomorphic to H is conjugate to H in $\text{Aut}(G)$.
3. There are $|\text{Out}(G)|/|\text{Out}(H)|$ conjugacy classes of subgroups isomorphic to H in G .

Proof: Since H is a maximal subgroup of G , $N_G(H) = H$. The corollary is now a direct consequence of Theorem 1.4 taking into account Remark 1.5 about the number of conjugacy classes of subgroups of G isomorphic to H . ■

2. Localization in alternating groups

We describe in this section a method for finding localizations of finite simple groups in alternating groups. Let H be a simple group and K a subgroup of index n . The (left) action of H on the cosets of K in H defines a **permutation representation** $H \rightarrow S_n$ as in [23, Theorem 3.14, p. 53]. The **degree** of the representation is the number n of cosets. As H is simple, this homomorphism is actually an inclusion $H \hookrightarrow A_n$. Recall that $\text{Aut}(A_n) = S_n$ if $n \geq 7$.

THEOREM 2.1: *Let H be a non-abelian finite simple group and K a maximal subgroup of index $n \geq 7$. Suppose that the following two conditions hold:*

1. *The order of K is maximal (among all maximal subgroups).*
2. *Any subgroup of H of index n is conjugate to K .*

Then the permutation representation $H \hookrightarrow A_n$ is a localization.

Proof: We show that the conditions of Theorem 1.4 are satisfied, starting with condition (1). Since K is maximal, it is self-normalizing and therefore the action of H on the cosets of K is isomorphic to the conjugation action of H on the set of conjugates of K . By our second assumption, this set is left invariant under $\text{Aut}(H)$. Thus the action of H extends to $\text{Aut}(H)$ and this yields the desired extension $\text{Aut}(H) \rightarrow S_n = \text{Aut}(A_n)$.

To check condition (2) of Theorem 1.4, let H' be a subgroup of A_n which is isomorphic to H and denote by $\alpha: H \rightarrow H'$ an isomorphism. Let J be the stabilizer of a point in $\{1, \dots, n\}$ under the action of H' . Since the orbit of this point has cardinality $\leq n$, the index of J is at most n , hence equal to n by our first assumption. Thus H' acts transitively. So H has a second transitive action via α and the action of H' . For this action, the stabilizer of a point is a subgroup of H of index n , hence conjugate to K by assumption. So K is also the stabilizer of a point for this second action and this shows that this action of H is isomorphic to the permutation action of H on the cosets of K , that is, to the first action. It follows that the permutation representation $H \xrightarrow{\alpha} H' \hookrightarrow A_n$ is conjugate in S_n to $H \hookrightarrow A_n$.

Finally, since H is a transitive subgroup of S_n with maximal stabilizer, the centralizer $C_{S_n}(H)$ is trivial by [7, Theorem 4.2A (vi)] and thus condition (3) of Theorem 1.4 is satisfied. ■

Among the twenty-six sporadic simple groups, twenty have a subgroup which satisfies the conditions of Theorem 2.1.

COROLLARY 2.2: *The following inclusions are localizations:*

$$\begin{aligned} M_{11} \hookrightarrow A_{11}, M_{22} \hookrightarrow A_{22}, M_{23} \hookrightarrow A_{23}, M_{24} \hookrightarrow A_{24}, J_1 \hookrightarrow A_{266}, J_2 \hookrightarrow A_{100}, \\ J_3 \hookrightarrow A_{6156}, J_4 \hookrightarrow A_{173067389}, HS \hookrightarrow A_{100}, McL \hookrightarrow A_{275}, Co_1 \hookrightarrow A_{98280}, \\ Co_2 \hookrightarrow A_{2300}, Co_3 \hookrightarrow A_{276}, Suz \hookrightarrow A_{1782}, He \hookrightarrow A_{2058}, Ru \hookrightarrow A_{4060}, \\ Fi_{22} \hookrightarrow A_{3510}, Fi_{23} \hookrightarrow A_{31671}, HN \hookrightarrow A_{1140000}, Ly \hookrightarrow A_{8835156}. \end{aligned}$$

Proof: In each case, it suffices to check in the ATLAS [5] that the conditions of Theorem 2.1 are satisfied. It is however necessary to check the complete list of maximal subgroups in [15] for the Fischer group Fi_{23} and [16] for the Janko group J_4 . ■

We obtain now two infinite families of localizations. The classical projective special linear groups $L_2(q) = PSL_2(q)$ of type $A_1(q)$, as well as the projective special unitary groups $U_3(q) = PSU_3(q)$ of type ${}^2A_2(q)$, are almost all connected to an alternating group by a localization. Recall that the notation $L_2(q)$ is used only for the simple projective special linear groups, that is if the prime power $q \geq 4$. Similarly, the notation $U_3(q)$ is used for $q > 2$.

PROPOSITION 2.3: (i) *The permutation representation $L_2(q) \hookrightarrow A_{q+1}$ induced by the action of $SL_2(q)$ on the projective line is a localization for any prime power $q \notin \{4, 5, 7, 9, 11\}$.*

(ii) *The permutation representation $U_3(q) \hookrightarrow A_{q^3+1}$ induced by the action of $SU_3(q)$ on the set of isotropic points in the projective plane is a localization for any prime power $q \neq 5$.*

Proof: We prove both statements at the same time. The group $L_2(q)$ acts on the projective line, whereas $U_3(q)$ acts on the set of isotropic points in the projective plane. In both cases, let B be the stabilizer of a point for this action (Borel subgroup). Let us also denote by G either $L_2(q)$ or $U_3(q)$, where q is a prime power as specified above, and r is $q + 1$, or $q^3 + 1$ respectively. Then B is a subgroup of G of index r by [13, Satz II-8.2] and [13, Satz II-10.12].

By [13, Satz II-8.28], which is an old theorem of Galois when q is a prime, the group $L_2(q)$ has no non-trivial permutation representation of degree less than r if $q \notin \{4, 5, 7, 9, 11\}$. The same holds for $U_3(q)$ by [6, Table 1] if $q \neq 5$. Thus B satisfies condition (1) of Theorem 2.1.

It remains to show that condition (2) is also satisfied. The subgroup B is the normalizer of a Sylow p -subgroup U , and $B = UT$, where T is a complement of U in B . If N denotes the normalizer of T in G , we know that $G = UNU$. This is the Bruhat decomposition (for more details see [3, Chapter 8]). We are now ready

to prove that any subgroup of G of index r is conjugate to B . Let H be such a subgroup. It contains a Sylow p -subgroup, and we can thus assume it actually contains U . Since G is generated by U and N , the subgroup H is generated by U and $N \cap H$. Assume H contains an element $x \in N - T$. The Weyl group N/T is cyclic of order two, generated by the class of x (the linear group $L_2(q)$ is a Chevalley group of type A_1 and the unitary group $U_3(q)$ is a twisted Chevalley group of type 2A_2). Moreover, $G = \langle U, xUx^{-1} \rangle$ by [12, Theorem 2.3.8 (e)] and both U and its conjugate xUx^{-1} are contained in H . This is impossible because $H \neq G$, so $N \cap H = T \cap H$. It follows that H is contained in $\langle U, T \rangle = B$. But H and B have the same order and therefore $H = B$. ■

Remark 2.4: This proof does not work for the action of $L_{n+1}(q)$ on the n -dimensional projective space if $n \geq 2$, because there is a second action of the same degree (on the set of all hyperplanes in $(\mathbb{F}_q)^{n+1}$). Thus there is another conjugacy class of subgroups of the same index, so condition (1) does not hold.

3. Proof of the main theorem

In order to prove our main theorem, we have to check that any group of the list is connected to an alternating group by a zigzag of localizations. When no specific proof is indicated for an inclusion to be a localization, it means that all the necessary information for checking conditions (1)–(3) of Theorem 1.4 is available in the ATLAS [5]. By C_2 we denote the cyclic group of order 2.

(i) *Alternating groups.*

The inclusions $A_n \hookrightarrow A_{n+1}$, for $n \geq 7$, studied by Libman in [19, Example 3.4] are localizations by Corollary 1.7, with $\text{Out}(A_n) \cong C_2 \cong \text{Out}(A_{n+1})$. The inclusion $A_5 \hookrightarrow A_6$ is a localization as well, since we have $\text{Out}(A_6) \cong (C_2)^2$, $\text{Out}(A_5) \cong C_2$, and there are indeed two conjugacy classes of subgroups of A_6 isomorphic to A_5 with fusion in $\text{Aut}(A_6)$. The inclusion $A_6 \hookrightarrow A_7$ is not a localization, but we can connect these two groups via a zigzag of localizations, for example as follows:

$$A_6 \hookrightarrow T \hookrightarrow Ru \hookleftarrow L_2(13) \hookrightarrow A_{14}$$

where T denotes the Tits group, Ru the Rudvalis group and the last arrow is a localization by Proposition 2.3.

(ii) *Chevalley groups $L_2(q)$.*

By Proposition 2.3, all but five linear groups $L_2(q)$ are connected to an alternating group. The groups $L_2(4)$ and $L_2(5)$ are isomorphic to A_5 , and $L_2(9) \cong A_6$.

We connect $L_2(7)$ to A_{28} via a chain of two localizations

$$L_2(7) \hookrightarrow U_3(3) = G_2(2)' \hookrightarrow A_{28}$$

where we use Theorem 2.1 for the second map. Similarly, we connect $L_2(11)$ to A_{22} via the Mathieu group M_{22} , using Corollary 2.2:

$$L_2(11) \hookrightarrow M_{22} \hookrightarrow A_{22}.$$

(iii) *Chevalley groups $U_3(q)$.*

For $q \neq 5$, we have seen in Proposition 2.3 (ii) that $U_3(q) \hookrightarrow A_{q^3+1}$ is a localization. One checks in [5, p. 34] that there is a localization $A_7 \hookrightarrow U_3(5)$, which connects $U_3(5)$ to the alternating groups.

(iv) *Chevalley groups $G_2(p)$.*

When p is an odd prime such that $(p+1, 3) = 1$, we will see in Proposition 4.3 that $U_3(p) \hookrightarrow G_2(p)$ is a localization. We can conclude by (iii), since 5 is not a prime in the considered family.

(v) *Sporadic simple groups.*

By Corollary 2.2, we already know that twenty sporadic simple groups are connected with some alternating group. We now show how to connect all the other sporadic groups, except the Monster for which we do not know what happens.

For the Mathieu group M_{12} , we note that the inclusion $M_{11} \hookrightarrow M_{12}$ is a localization because there are two conjugacy classes of subgroups of M_{12} isomorphic to M_{11} (of index 11) with fusion in $\text{Aut}(M_{12})$ (cf. [5, p. 33]). We conclude by Corollary 1.7.

The list of all maximal subgroups of Fi'_{24} is given in [21] and one applies Theorem 1.4 to show that $He \hookrightarrow Fi'_{24}$ is a localization (both groups have C_2 as group of outer automorphisms).

Looking at the complete list of maximal subgroups of the Baby Monster B in [27], we see that $Fi_{23} \hookrightarrow B$ is a localization, as well as $Th \hookrightarrow B$, $HN \hookrightarrow B$, and $L_2(11) \hookrightarrow B$ (see Proposition 4.1 in [27]). This connects Thompson's group Th and the Baby Monster (as well as the Harada-Norton group HN) to the Fischer groups and also to the Chevalley groups $L_2(q)$.

Finally we consider the O'Nan group $O'N$. By [26, Proposition 3.9] we see that $M_{11} \hookrightarrow O'N$ satisfies conditions (1)–(3) of Corollary 1.7 and thus is a localization.

(vi) *Other Chevalley groups.*

The construction of the sporadic group Suz provides a sequence of five graphs (the Suzuki chain) whose groups of automorphisms are successively $\text{Aut}(L_2(7))$,

$\text{Aut}(G_2(2)'), \text{Aut}(J_2), \text{Aut}(G_2(4))$ and $\text{Aut}(Suz)$ (see [11, pp. 108–9]). Each one of these five groups is an extension of C_2 by the appropriate finite simple group. All arrows in the sequence

$$(3.1) \quad L_2(7) \hookrightarrow G_2(2)' \hookrightarrow J_2 \hookrightarrow G_2(4) \hookrightarrow Suz$$

are thus localizations by Corollary 1.7 because they are actually inclusions of the largest maximal subgroup (cf. [5]). This connects the groups $G_2(2)'$ and $G_2(4)$ to alternating groups since we already know that Suz is connected to A_{1782} by Corollary 2.2. Alternatively, note that $G_2(4) \hookrightarrow L_2(13) \hookrightarrow A_{14}$ are localizations, using Proposition 2.3 for the second one.

The Suzuki group provides some more examples of localizations: $L_3(3) \hookrightarrow Suz$ by [25, Section 6.6], and $U_5(2) \hookrightarrow Suz$ by [25, Section 6.1]. We also have localizations

$$A_9 \hookrightarrow D_4(2) = O_8^+(2) \hookrightarrow F_4(2) \hookrightarrow {}^3D_4(2)$$

which connect these Chevalley groups (see Proposition 4.4 for the last arrow). We are able to connect three symplectic groups since $A_8 \hookrightarrow S_6(2)$ and $S_4(4) \hookrightarrow He$ are localizations, as well as $S_8(2) \hookrightarrow A_{120}$ by Theorem 2.1. This allows us in turn to connect more Chevalley groups as $U_4(2) \hookrightarrow S_6(2)$, and $O_8^- = {}^2D_4(2) \hookrightarrow S_8(2)$ are both localizations.

Each of the following localizations involves a linear group and connects some new group to the component of the alternating groups:

$$L_2(11) \hookrightarrow U_5(2), L_3(3) \hookrightarrow T, L_2(7) \hookrightarrow L_3(11), \text{ and } L_4(3) \hookrightarrow F_4(2).$$

The localization $U_3(3) \hookrightarrow G_2(5)$ connects $G_2(5)$ and thus $L_3(5)$ by Proposition 4.2 below. Likewise, since we just showed above that $L_3(11)$ belongs to the same rigid component, then so does $G_2(11)$.

Next $M_{22} \hookrightarrow U_6(2)$ and $A_{12} \hookrightarrow O_{10}^- = {}^2D_5(2)$ are also localizations.

In the last three localizations, connecting the groups $U_4(3), E_6(4),$ and $D_4(3)$, the order of the outer automorphism groups is larger than 2. Nevertheless, Theorem 1.4 applies easily. There is a localization $A_7 \hookrightarrow U_4(3)$. There are four conjugacy classes of subgroups of $U_4(3)$ isomorphic to A_7 , all of them being maximal. The dihedral group $D_8 \cong \text{Out}(U_4(3))$ acts transitively on those classes and S_7 is contained in $\text{Aut}(U_4(3))$ (see [5, p. 52]).

We have also a localization $J_3 \hookrightarrow E_6(4)$. Here $\text{Out}(E_6(4)) \cong D_{12}$ and there are exactly six conjugacy classes of subgroups isomorphic to J_3 in $E_6(4)$ which are permuted transitively by D_{12} . This is precisely the statement of the main theorem of [17].

Finally $D_4(2) \hookrightarrow D_4(3)$ is a localization. Here we have $\text{Out}(D_4(2)) \cong S_3$ and $\text{Out}(D_4(3)) \cong S_4$. There are four conjugacy classes of subgroups of $D_4(3)$ isomorphic to $D_4(2)$.

4. Other localizations

In this section, we give further examples of localizations between simple groups. We start with three infinite families of localizations. Except the second family, we do not know if the groups belong to the rigid component of alternating groups.

LEMMA 4.1: *Let p be an odd prime with $(3, p - 1) = 1$. If $H \leq M \leq G_2(p)$ are subgroups with $H \cong L_3(p)$ and M maximal, then $M \cong \text{Aut}(L_3(p))$.*

Proof: The order of $L_3(p)$ is $p^3(p^3 - 1)(p^2 - 1)$, which is larger than p^6 . So the main theorem in [20] implies that any maximal subgroup of $G_2(p)$ containing H has to be one of [20, Table 1, p. 300] or a parabolic subgroup. Of course $|L_3(p)|$ has to divide $|M|$. Therefore M cannot be parabolic because the order of a parabolic subgroup of $G_2(p)$ is $p^6(p^2 - 1)(p - 1)$ (see [14, Theorem A]). When $p \neq 3$ there are only two maximal subgroups left to deal with. When $p = 3$, we could also have used the ATLAS [5, p. 60]. In both cases the only maximal subgroup whose order is divisible by $|L_3(p)|$ is isomorphic to $\text{Aut}(L_3(p))$. ■

PROPOSITION 4.2: *Let p be an odd prime with $(3, p - 1) = 1$. Then there is a localization $L_3(p) \hookrightarrow G_2(p)$.*

Proof: The strategy is to verify conditions (1)–(3) of Theorem 1.4, or rather conditions (1), (2), and (3') of Remark 1.5. Let us first review some facts from [14, Proposition 2.2]. There exist two subgroups $L_+ \leq K_+ \leq G_2(p)$ with $L_+ \cong L_3(p)$ a subgroup of index 2 in K_+ . This is the inclusion we consider here. Moreover $K_+ = N_{G_2(p)}(L_+)$ (this is Step 3 in [14, Proposition 2.2]), and $N_{\text{Aut}(G_2(p))}(L_+) \cong \text{Aut}(L_3(p))$ (Step 1). Thus $K_+ \cong \text{Aut}(L_3(p))$ and condition (1) of Theorem 1.4 obviously holds. To check condition (2) consider a subgroup $H \leq G_2(p)$ with $H \cong L_3(p)$. By the above lemma H must be contained in a maximal subgroup isomorphic to $\text{Aut}(L_3(p))$. When $p \neq 3$ the group $G_2(p)$ is complete and [14, Theorem A] shows that H is contained in some conjugate of K_+ . Therefore H is conjugate to L_+ in $G_2(p)$. When $p = 3$, the group $G_2(p)$ is not complete. The group $\text{Out}(G_2(3))$ is cyclic of order 2 as there is a “graph automorphism” $\gamma: G_2(3) \rightarrow G_2(3)$ which is not an inner automorphism (we follow the notation from [14, Proposition 2.2 (v)]). Then K_+ and $\gamma(K_+)$ are not conjugate in $G_2(3)$,

but of course they are in $\text{Aut}(G_2(3))$. Again Theorem A in [14] shows that H is conjugate either to L_+ or to $\gamma(L_+)$ in $G_2(3)$ and condition (2) is also satisfied. Finally condition (3') is valid since $\text{Out}(L_3(p))$ is cyclic of order 2. The number of conjugacy classes of subgroups isomorphic to $L_3(p)$ in $G_2(p)$ is 1 when $p \neq 3$ and 2 when $p = 3$. ■

PROPOSITION 4.3: *Let p be an odd prime with $(3, p + 1) = 1$. Then there is a localization $U_3(p) \hookrightarrow G_2(p)$.*

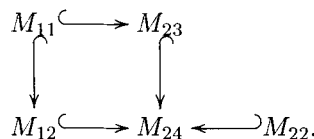
Proof: The proof is similar to that of the preceding proposition. Apply also [14, Proposition 2.2] for the subgroup K_- . ■

PROPOSITION 4.4: *There is a localization ${}^3D_4(p) \hookrightarrow F_4(p)$ for any prime p .*

Proof: We have $\text{Out}(F_4(2)) \cong C_2$ while, for an odd prime p , $F_4(p)$ is complete. On the other hand, $\text{Out}({}^3D_4(p))$ is cyclic of order 3. By [20, Proposition 7.2] there are exactly $(2, p)$ conjugacy classes of subgroups isomorphic to ${}^3D_4(p)$ in $F_4(p)$, fused by an automorphism if $p = 2$. Applying Theorem 1.4, we see that the inclusion ${}^3D_4(p) \hookrightarrow \text{Aut}({}^3D_4(p)) \hookrightarrow F_4(p)$ given by [20, Table 1, p. 300] is a localization. ■

We have seen various localizations involving sporadic groups in the proof of the main theorem. We give here further examples.

We start with the five Mathieu groups. Recall that the Mathieu groups M_{12} and M_{22} have C_2 as outer automorphism groups, while the three other Mathieu groups are complete. The inclusions $M_{11} \hookrightarrow M_{23}$ and $M_{23} \hookrightarrow M_{24}$ are localizations by Corollary 1.6. We have already seen in Section 3 that the inclusion $M_{11} \hookrightarrow M_{12}$ is a localization. The inclusion $M_{12} \hookrightarrow M_{24}$ is also a localization. Indeed $\text{Aut}(M_{12})$ is the stabilizer in M_{24} of a pair of dodecads; the stabilizer of a single dodecad is a copy of M_{12} . Up to conjugacy, these are the only subgroups of M_{24} isomorphic to M_{12} and thus the condition (3') in Remark 1.5 is satisfied. Similarly, $M_{22} \hookrightarrow M_{24}$ is also a localization, because $\text{Aut}(M_{22})$ can be identified as the stabilizer of a duad (a pair of octads) in M_{24} whereas M_{22} is the pointwise stabilizer (see [5, p. 39 and p. 94]). In short we have the following diagram, where all inclusions are localizations:



We consider next the sporadic groups linked to the Conway group Co_1 . Inside Co_1 sits Co_2 as stabilizer of a certain vector OA of type 2 and Co_3 as stabilizer of another vector OB of type 3. These vectors are part of a triangle OAB and its stabilizer is the group HS , whereas its setwise stabilizer is $Aut(HS)$. The Conway groups are complete, the smaller ones are maximal simple subgroups of Co_1 and there is a unique conjugacy class of each of them in Co_1 as indicated in the ATLAS [5, p. 180]. Hence $Co_2 \hookrightarrow Co_1$ and $Co_3 \hookrightarrow Co_1$ are localizations by Corollary 1.6. Likewise the inclusions $HS \hookrightarrow Co_2$ and $McL \hookrightarrow Co_3$ are also localizations: They factor through their group of automorphisms, since, for example, $Aut(McL)$ is the setwise stabilizer of a triangle of type 223 in the Leech lattice, a vertex of which is stabilized by Co_3 . Finally, $M_{22} \hookrightarrow HS$ is a localization for similar reasons, since any automorphism of M_{22} can be seen as an automorphism of the Higman–Sims graph (cf. [1, Theorem 8.7 p. 273]). We get here the following diagram of localizations:

$$\begin{array}{ccccc}
 M_{22} & & McL & \hookrightarrow & Co_3 \\
 \downarrow & & & & \downarrow \\
 HS & \hookrightarrow & Co_2 & \hookrightarrow & Co_1
 \end{array}$$

Some other related localizations are $M_{23} \hookrightarrow Co_3$, $M_{23} \hookrightarrow Co_2$ and $M_{11} \hookrightarrow HS$.

We move now to the Fischer groups and Janko’s group J_4 . The inclusion $T \hookrightarrow Fi_{22}$ is a localization (both have C_2 as outer automorphism groups) as well as $M_{12} \hookrightarrow Fi_{22}$, and $A_{10} \hookrightarrow Fi_{22}$. Associated to the second Fischer group, we have a chain of localizations

$$A_{10} \hookrightarrow S_8(2) \hookrightarrow Fi_{23}.$$

By [15, Theorem 1] the inclusion $A_{12} \hookrightarrow Fi_{23}$ is also a localization. Moreover, $M_{11} \hookrightarrow J_4$ and $M_{23} \hookrightarrow J_4$ are localizations by Corollaries 6.3.2 and 6.3.4 in [16].

Let us list now without proofs a few inclusions we know to be localizations. We start with two examples of localizations of alternating groups: $A_{12} \hookrightarrow HN$, and $A_7 \hookrightarrow Suz$ by [25, Section 4.4]. To conclude, we list a few localizations of Chevalley groups: $L_2(8) \hookrightarrow S_6(2)$, $L_2(13) \hookrightarrow G_2(3)$, $L_2(32) \hookrightarrow J_4$ (by [16, Proposition 5.3.1]), $U_3(3) \hookrightarrow S_6(2)$, ${}^3D_4(2) \hookrightarrow Th$, $G_2(5) \hookrightarrow Ly$, $E_6(2) \hookrightarrow E_7(2)$, and $E_6(3) \hookrightarrow E_7(3)$. The inclusion $E_6(q) \hookrightarrow E_7(q)$ is actually a localization if and only if $q = 2$ or $q = 3$ by [20, Table 1]. The main theorem in [24] states that $Sz(32) \hookrightarrow E_8(5)$ is a localization. There is a single conjugacy class of subgroups isomorphic to $Sz(32)$ in $E_8(5)$, and $Out(Sz(32))$ is cyclic of order 5.

We have seen a great deal of localizations of finite simple groups, and one could think at this point that they abound in nature. This is, of course, not so: During our work on this paper, we came across many more inclusions of simple groups that are not localizations. They do not appear here for obvious reasons. On the other hand, our list is certainly far from being complete. It would be nice, for example, to find other infinite families of localizations among groups of Lie type and to determine if they are connected to the alternating groups. Another interesting task would be to find out which simple groups are local with respect to a given localization $i: H \hookrightarrow G$, or, even better, to compute the localization of other simple groups with respect to i . Will they still be finite simple groups?

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